CIMT Further Stats p18 Ex 1·11

1. X = length of cloth wasted

$$f(x) = \begin{cases} \frac{1}{a} & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$ ie. $X \sim U(0,a)$ uniform continuous distribution.

a) $E(X) = \int_0^a x f(x)\,dx$

$= \int_0^a x \cdot \frac{1}{a} \cdot dx$

$= \frac{1}{a}\int_0^a x \, dx$

$= \frac{1}{a}\left[\frac{1}{2}x^2\right]_0^a$

$= \frac{1}{2a}\left[x^2\right]_0^a$

$= \frac{1}{2a}(a^2 - 0^2)$

$= \frac{a^2}{2a}$

$= \underline{\underline{\frac{a}{2}}}$.

$E(X^2) = \int_0^a x^2 f(x)\,dx$

$= \int_0^a x^2 \cdot \frac{1}{a} \cdot dx$

$= \frac{1}{a}\int_0^a x^2\,dx$

$= \frac{1}{a}\left[\frac{1}{3}x^3\right]_0^a$

$= \frac{1}{3a}\left[x^3\right]_0^a$

$= \frac{1}{3a}(a^3 - 0^3)$

$= \frac{a^3}{3a}$

$= \frac{a^2}{3}$

$Var(X) = E(X^2) - E^2(X)$

$= \frac{a^2}{3} - \left(\frac{a}{2}\right)^2$

$= \frac{a^2}{3} - \frac{a^2}{4}$

$= \frac{4a^2}{12} - \frac{3a^2}{12}$

$= \underline{\underline{\frac{a^2}{12}}}$

b) let $\bar{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$

$E(\bar{X}) = E\left(\frac{1}{n}(X_1 + \ldots + X_n)\right)$

$= \frac{1}{n}\left(E(X_1 + \ldots X_n)\right)$

$= \frac{1}{n}\left(E(X_1) + \ldots + E(X_n)\right)$

$= \frac{1}{n}\left(n \times \frac{a}{2}\right)$

$= \underline{\underline{\frac{a}{2}}}$.

$Var(\bar{X}) = Var\left(\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right)$

$= \left(\frac{1}{n}\right)^2 Var(X_1 + \ldots + X_n)$

$= \left(\frac{1}{n}\right)^2 \cdot \left(Var(X_1) + \ldots + Var(X_n)\right)$

$= \frac{1}{n^2} \cdot \left(n \times \frac{a^2}{12}\right)$

$= \underline{\underline{\frac{a^2}{12n}}}$

c) Y = median of random sample of size 3

$$g(y) = \begin{cases} \frac{6y}{a^2} - \frac{6y^2}{a^3} & 0<y<a \\ 0 & \text{otherwise} \end{cases}$$

$E(Y) = \int_0^a y \cdot g(y)\,dy$

$= \int_0^a y \cdot \left(\frac{6y}{a^2} - \frac{6y^2}{a^3}\right) dy$

$= \frac{6}{a^3}\int_0^a y(ay - y^2)\,dy$

$= \frac{6}{a^3}\int_0^a (ay^2 - y^3)\,dy$

$$= \frac{6}{a^3}\left[\frac{1}{3}ay^3 - \frac{1}{4}y^4\right]_0^a$$

$$= \frac{6}{a^3}\left(\left(\frac{1}{3}.a.a^3 - \frac{1}{4}.a^4\right) - (0-0)\right)$$

$$= \frac{6}{a^3}\left(\frac{a^4}{3} - \frac{a^4}{4}\right)$$

$$= \frac{6}{a^3}\cdot\frac{a^4}{12}$$

$$= \frac{a}{2}.$$

$$E(Y^2) = \int_0^a y^2 g(y)\,dy$$

$$= \frac{6}{a^3}\int_0^a (ay^3 - y^4)\,dy$$

$$= \frac{6}{a^3}\left[\frac{1}{4}ay^4 - \frac{1}{5}y^5\right]_0^a$$

$$= \frac{6}{a^3}\left(\left(\frac{1}{4}.a.a^4 - \frac{1}{5}.a^5\right) - (0-0)\right)$$

$$= \frac{6}{a^3}\left(\frac{a^5}{4} - \frac{a^5}{5}\right)$$

$$= \frac{6}{a^3}\cdot\frac{a^5}{20}$$

$$= \frac{3a^2}{10}$$

so $Var(Y) = E(Y^2) - E^2(Y)$

$$= \frac{3a^2}{10} - \left(\frac{a}{2}\right)^2$$

$$= \frac{3a^2}{10} - \frac{a^2}{4}$$

$$= \frac{6a^2}{20} - \frac{5a^2}{20}$$

$$= \frac{a^2}{20}$$

so Y has mean $\frac{a}{2}$ and variance $\frac{a^2}{20}$.

3. X = thickness of hardboard

$X \sim N(7.3, 0.5^2)$

a) $P(7 < X < 8) = P\left(\frac{7-7.3}{0.5} < Z < \frac{8-7.3}{0.5}\right)$

$= P(-0.6 < Z < 1.4)$

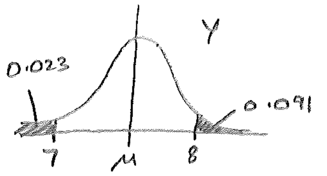
$\approx 0.64499\ldots$

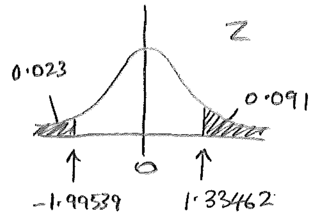
≈ 0.6450 (4dp)

b) let $Y \sim N(\mu, \sigma^2)$

$P(Y > 8) = 0.091$

$P(Y < 7) = 0.023$





so $\frac{7-\mu}{\sigma} = -1.99539$

and $\frac{8-\mu}{\sigma} = 1.33462$

solve simultaneously using LinSolve command

to give $\mu = 7.59921\ldots$

$\sigma = 0.300299\ldots$

so, to nearest tenth, $\mu = 7.6$ mm

$\sigma = 0.3$ mm.

c) let $W = Y - X$

$W \sim N(7.6 - 7.3, 0.3^2 + 0.5^2)$

$W \sim N(0.3, 0.34)$

$P(X > Y) = P(0 > Y - X)$

$= P(Y - X < 0)$

$= P(W < 0)$

$= P\left(Z < \frac{0-0.3}{\sqrt{0.34}}\right)$

$= P(Z < -0.514496)$

$= 0.303453\ldots$

≈ 0.3035 (4dp)

d) i) minimise proportion would be to have mean at midpoint between 7 and 8, by symmetry

ie. $\mu = 7.5$mm

ii) let V = thickness of hardboard

$V \sim N(\mu, 0.5^2)$

we want $P(V < 7) = 0.001$

so $P\left(Z < \frac{7-\mu}{0.5}\right) = 0.001$

$\frac{7-\mu}{0.5} = -3.09023$ using invNorm(0.001)

$\mu = 7 + 0.5 \times 3.09023$

$\mu = 8.54512\ldots$

$\mu \doteqdot 8.55$ mm. (2dp)

4. X = mass of beans in one tin

$X \sim N(425, 25^2)$

T = mass of tin

$T \sim N(90, 10^2)$

a) i) $P(X+T > 550) = ?$

let $W = X+T$

$W \sim N(425+90, 25^2+10^2)$

$W \sim N(515, 725)$

so $P(W > 550) = P\left(Z > \frac{550-515}{\sqrt{725}}\right)$

$= P(Z > 1.29987)$

$\approx 0.096823\ldots$

$\approx \underline{0.0968}$ (4dp)

ii) $P(466 < X+T < 575)$

$= P(466 < W < 575)$

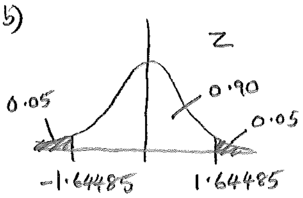
$= P\left(\frac{466-515}{\sqrt{725}} < Z < \frac{575-515}{\sqrt{725}}\right)$

$= P(-1.81981 < Z < 2.22834)$

$= 0.952677\ldots$

$\approx \underline{0.9527}$ (4dp)

b)



using invNorm

so filled tins will lie within ± 1.64485 st. deviations of the mean

ie. $515 \pm 1.64485 \times \sqrt{725}$

515 ± 44.289

from $\underline{470.71 \text{ to } 559.29}$ (2dp)

c) let $V = W_1 + \ldots + W_{24} + B$ where $B \sim N(500, 30^2)$ and B = mass of box.

so $E(V) = 24 \times 515 + 500$

$= 12860$

$Var(V) = 24 \times 725 + 30^2$

$= 18300$

so $P(V < 12750) = P\left(Z < \frac{12750 - 12860}{\sqrt{18300}}\right)$

$= P\left(Z < \frac{-110}{\sqrt{18300}}\right)$

$= P(Z < -0.813143\ldots)$

$= 0.208068\ldots$

$\approx \underline{0.2081}$ (4dp)

7. nominal mass of fertilizer = 12kg.

B = mass of bag of fertiliser

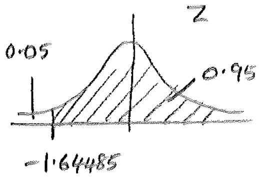
$B \sim N(12.05, 0.2^2)$

a) $P(B > 12) = P\left(Z > \frac{12 - 12.05}{0.2}\right)$

$= P(Z > -0.25)$

$\doteq 0.5987$ (4dp)

b) $P(B > b) = 0.95$



$P\left(Z > \frac{b - 12.05}{0.2}\right) = 0.95$

$\frac{b - 12.05}{0.2} = -1.64485$

$b = 12.05 - 0.2 \times 1.64485$

$b = 11.721$

$b \approx 11.72$ kg. (2dp)

c) let m = mean mass of 20 bags.

so $m = \frac{1}{20}(B_1 + \dots + B_{20})$

$E(m) = E\left(\frac{1}{20}(B_1 + \dots + B_{20})\right)$

$= \frac{1}{20} E(B_1 + \dots + B_{20})$

$= \frac{1}{20}\left(E(B_1) + \dots + E(B_{20})\right)$

$= \frac{1}{20} \cdot 20 \times 12.05$

$= 12.05$

$Var(m) = Var\left(\frac{1}{20}(B_1 + \dots + B_{20})\right)$

$= \left(\frac{1}{20}\right)^2 Var(B_1 + \dots + B_{20})$

$= \frac{1}{20^2} \times 20 \times 0.2^2$

$= \frac{0.04}{20}$

$= 0.002$

so $m \sim N(12.05, 0.002)$

$P(m > 12) = P\left(Z > \frac{12 - 12.05}{\sqrt{0.002}}\right)$

$= P(Z > -1.11803\dots)$

$= 0.868224\dots$

≈ 0.8682 (4dp)

d)

let $T = B_1 + \dots + B_{20}$, where T = total weight of 20 bags.

so $E(T) = 20 \times 12.05$

$= 241$

$Var(T) = 20 \times 0.2^2$

$= 0.8$

so $\underline{T \sim N(241, 0.8)}$

$$P(239.5 < T < 240.5) = P\left(\frac{239.5 - 241}{\sqrt{0.8}} < Z < \frac{240.5 - 241}{\sqrt{0.8}}\right)$$

$$= P(-1.67705 < Z < -0.559017)$$

$$= 0.241309\dots$$

$$\approx \underline{\underline{0.2413}} \text{ (4dp)}$$

NEW MACHINE !

$C_i \sim N(12.05, 0.05)$

let $U = B_1 + \dots + B_n + C_1 + \dots + C_{20-n}$ (ie. n bags from 1st machine, $20-n$ bags from 2nd machine) and U = total mass of 20 bags from these machines.

$E(U) = E(B_1 + \dots + B_n + C_1 + \dots + C_{20-n})$

$= n \times 12.05 + (20-n) \times 12.05$

$= 20 \times 12.05$

$= 241$

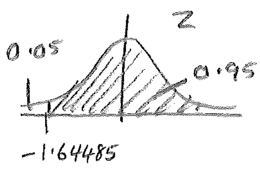
$Var(U) = Var(B_1 + \dots + B_n + C_1 + \dots + C_{20-n})$

$= n \times 0.2^2 + (20-n) \times 0.05^2$

$= 0.05 + 0.0375n \quad (= \sigma^2)$

now $P(U > 240) > 0.95$

$P\left(Z > \frac{240 - 241}{\sigma}\right) > 0.95$



$\frac{240 - 241}{\sigma} > -1.64485$

$\frac{-1}{\sigma} > -1.64485$

$\frac{1}{\sigma} < 1.64485$

$\sigma > \frac{1}{1.64485}$

$\sqrt{0.05 + 0.0375n} > \frac{1}{1.64485}$

$0.05 + 0.0375n > 0.369612$

$n > 8.52297$

so $\underline{n \text{ must be } 9.}$, as $n \in \mathbb{Z}^+$!